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The Bounded Logit & Probit Estimators: As Implemented in the DISCRETE program.

The DOUBLE model of the DISCRETE program supports a variety of "bounded" logit and probit models (often referred to as dichotomous choice models). Bounded means that one, or several, "bounds" values are available.

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1. Synopsis

Typically, these bounds are "bids" that a respondent is presented with. For example, a respondent may be asked whether he would choose a project with attributes X, if it would cost him Y dollars. Or, the respondent would answer a sequence of such questions, so that the analyst knows the two bids between which the respondent switches from a YES to a NO.

DISCRETE uses the DOUBLE model to implement this class of estimators. Both a logit, weibit, and probit versions are supported. In addition, bivariate probit models are supported.

This document describes the models. For detailed instructions on how to estimate these models, see DISCRETE.TXT.

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2. The Basic Model

- We suppose n=1..N *individuals*.
- Individuals are characterized by a vector of explanatory variables (X).
- Each individual responds YES, or NO, to one or several different *bid* values (BID).
- An individual answers yes if his willingness to pay (WTP) exceeds the bid. :
- WTP is modeled as a function of X: WTP= $\mu + \epsilon$

Where, $\mu = X\beta$

- The observed answer (Y):
  - Y=1 iff WTP > BID
  - = 0 if WTP < BID (or equal to BID)

If ϵ has a logistic distribution, with scale parameter τ

$$PROB(NO \mid Bid) = 1 \big/ 1 + e^{(\mu - BID)/\tau}$$

If ϵ has a normal distribution with parameter σ,

$$PROB(NO \mid Bid) = \Phi((Bid - \mu)/\sigma)$$

If ϵ has a weibull (more precisely, an extreme value) distribution with parameter τ

$$PROB(NO \mid Bid) = \exp\left(-e^{(\mu - Bid)/\sigma}\right)$$

Notes:

- PROB(YES|*BID*) is simply 1-PROB(NO|*BID*).
- Note that sometimes (μ-*BID*) is used, other times (*BID*-μ) is used.
- Often, these models are presented as:
  $X\beta/\sigma$ - *BID*/σ; (or *BID*/σ - $X\beta/\sigma$)
  In which case, estimation provides estimates of $1/\sigma$ (the parameter on the *BID*), and β/σ. β and σ are then solved for, and standard errors are computed using one of several methods. DISCRETE does not use this shortcut-- σ and β (or τ) are directly estimated.
- Variances of ϵ are:
  Logistic: $\tau^2\pi$ / 3
  Weibull: $\tau^2\pi$ / 6
  Normal: σ^2

3a. The Single Bounded Model

In the single bounded model, the respondent is given just one bid to respond to.

The likelihood is:

$$PROB = PROB(NO \mid \mu, Bid)^{Y=0} + \left(1 - PROB(NO \mid \mu, Bid)\right)^{Y=1}$$

where Y=0 refers to the NO respondents, and Y=1 to the YES respondents.

3b. The Double Bounded Model

In the double bounded model, there are 3 generic responses:

Y = Yes to a high bid
YN = Yes to a low bid, NO to a high bid.
N = No to a low bid.

Notes:

- For Y responses, there typically is a "lower bid" that the respondent also answered YES to. For N responses, there typically is a "higher bid" that the respondent also answered NO to. The YN responses typically come from one of two response patterns:
  - YN : responsent answers YES to a starting bid and NO to a higher bid.
  - NY : respondent answers NO to a starting bid, and then YES to a lower bid;

  In both cases, the essential information is the same: a lower bound bid and a higher bound bid are known.

For the Y responses, an upper bid (*bidU*) is used; for N responses a lower bid is used (*bidL*), and for YN responses both (*bidU* and *bidL*) are used.

Each observation is assigned to one of 3 groups (each group contains all observations with a given generic response). The probability for each member of these groups is:

P_y	1- PROB(NO|μ,*bidU*)	*1- probability of saying No to upper bid*
P_n	PROB(NO|μ,*bidL*)	*probability of saying No to lower bid*
P_{yn}	PROB(NO|μ,*bidU*) - PROB(NO|μ,*bidL*)	*probability of saying No to upper bid minus probability of saying No to lower bid*

Where:

$\mu = X \beta$

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4. Specifying bids

DISCRETE supports several methods of specifying double bounded bids:

1. EXPLICIT. The upper and lower bids are available in two separate variables
2. INDICATOR: Three bid variables, and a CHOICE_INDICATOR variable are available. The bid variables contain values for a starting bid, a low followup bid, and a high followup bid. The CHOICE_INDICATOR variable signals which of the four following response patterns was chosen.
   i. YY = Yes to starting bid, Yes to high followup
   ii. YN = Yes to starting bid, No to high followup
   iii. NY == No to starting bid, yes to low followup
   iv. NN == No to starting bid, no to low followup
3. DUMMY: Three bid variables, and a set of 4 dummy indicator variables. Each dummy corresponds to one of the 4 response patterns (noted above). One, and only one, of the dummies should have a non-zero value.
4. MULTIPLE: M bid variable and M response dummies. The M bid variables contain increasing bid values. The response dummies equal 1 if a YES was given for the corresponding (m'th) bid value; and equals 0 if a NO was given.

For the details; see the description of the `CHOICE TYPE=x` keyphrase (in DESCRIBE.TXT).

5a. The bivariate model

A bivariate probit model is supported, each of which has several variants. In this model, -- a separate ϵ exists for the first and second responses.

- The `–BIVAR` option (on the `MODEL` keyword) is used to select which variant of the (single or double bounded) bivaraite model to estimate.

The likelihood for the double bounded bivariate probit is defined for each of the four response patterns. Note that, for the double bounded bivariate probit, the YN and NY response patterns must be separately identified (in contrast to the non-bivariate models, where they are combined).

$PROB(YES, YES \mid Bid_1, Bid_2, X)$	$\left(1-\Phi\left(\frac{Bid_2-\mu_2}{\sigma_2}\right)\right)+$ $\Phi_2(Bid_1-\mu_1, Bid_2-\mu_2, \sigma_1, \sigma_2, \rho)-$ $\Phi\left(\frac{Bid_1-\mu_1)}{\sigma_1}\right)$	BID_1 is the starting bid, BID_2 is the higher bid.
$PROB(YES, NO \mid Bid_1, Bid_2, X)$	$\Phi\left(\frac{Bid_2-\mu_2}{\sigma_2}\right)-$ $\Phi_2(Bid_1-\mu_1, Bid_2-\mu_2, \sigma_1, \sigma_2, \rho)$	BID_1 is the starting bid, BID_2 is the higher bid.
$PROB(NO, YES \mid Bid_1, Bid_2, X)$	$\Phi\left(\frac{Bid_1-\mu_1}{\sigma_1}\right)-$ $\Phi_2(Bid_1-\mu_1, Bid_2-\mu_2, \sigma_1, \sigma_2, \rho)$	BID_1 is the starting bid, BID_2 is the lower bid.

$PROB(NO, NO \mid Bid_1, Bid_2, X)$	$\Phi_2(Bid_1 - \mu_1, Bid_2 - \mu_2, \sigma_1, \sigma_2, \rho)$	BID_1 is the starting bid, BID_2 is the lower bid.

In the above:

- Φ is the standard normal CDF.
- Φ_2 is the bivariate normal CDF (with 0 means)
- $\mu_1 = X\beta_1$
- $\mu_2 = X\beta_2$

5b. 2-Stage single (or double) bounded probit

The likelihood for the single or double bounded 2-stage probit is defined across two stages: a participation stage, and a bid-response stage. For non-participants (who answer NO in the first stage), there is no second-stage information available.

- The `-2STAGE` option (on the `MODEL` keyword) is used to specify that a 2-stage (single or double bounded) model should be estimated.

The possible response patterns are:

N: Non-participants (no to first stage). There is no second stage information for these observations.

And depending on whether a single or double bounded model is estimated:

Single bounded:

YN: Participants who said No to a second stage bid value.

YY: Participants who said Yes to a second stage bid value.

Double bounded:

Y**YN**: Participants who said Yes to a lower bid, and No to a higher bid

Y**YY**: Participants who said Yes to a higher bid

Y**NN**: Participants who said No to a lower bid

The likelihoods for these response patterns:

	Single Bounded
$PROB(NO)$	$1-\Phi(\mu_1)$
$PROB(YES, NO)$	$\Phi(\mu) - \Phi_2(\mu_1, \mu_2 - Bid, 1, \sigma, \rho)$
$PROB(YES, YES)$	$\Phi_2(\mu_1, \mu_2 - Bid, 1, \sigma, \rho)$

	Double Bounded
$PROB(NO)$	$1-\Phi(\mu_1)$
$PROB(Yes, YES, NO)$	$\Phi_2(\mu_1, \mu_2 - BidL, 1, \sigma, \rho) - \Phi_2(\mu_1, \mu_2 - BidU, 1, \sigma, \rho)$

$PROB(Yes, YES, YES)$	$\Phi_2(\mu_1, \mu_2 - BidU, 1, \sigma, \rho)$
$PROB(Yes, NO, NO)$	$\Phi(\mu) - \Phi_2(\mu_1, \mu_2 - BidL, 1, \sigma, \rho)$

In the above:

- we use (μ - *bid*).
- Φ is the standard normal CDF.
- Φ_2 (x1,x2,sd1,sd2,rho) -- is the bivariate normal CDF (with 0 means); evaluated at x1 and x2, with standard deviation sd1 and sd2, and correlation coefficient rho.
- $\mu_1 = X_1\beta_1$
- $\mu_2 = X_2\beta_2$
- *bid, bidU, and bidL* – single bounded bid; upper and lower double bounded bids
- In contrast to the bivariate probit model, separate X variables (with separate coefficients) are used in each stage (you can specify the same X variables for each stage, but the coefficients will be different).

Note that when specifying a 2-stage model, you must specify a first stage response variable (using the `RESP1` keyword), and first stage independent variable (using the `X1` keyword). See DISCRETE.TXT for the details.

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6. Recoding

Using the `BOUND` and `NEVEYES` keywords, you can "recode" the bounds (the upper and lower bids) for each observation. This is particularly useful to deal with truncation at zero, and to account for additional information on particpation status.

`BOUND` is used to recode upper and lower bounds.

Basically, `BOUND` is used to convert YY and NN responses to YN responses; or to convert NY (or YN) to YY and NN responses.

For YY and NN responses:

When a YY (or NN) response is given (or in the single bound model), the upper (lower) bid value is impliticly equal to infinity (-infinity).
For example (using the Probit likelihood):

- the probability of a YY is computed by integrating from (*bidH*- *X*β) to infinity.
- the probability of a NN is computed by integrating from -infinity to (*bidL* - *X*β)

In many cases, these bounds are not reasonable. In particular, the implicit use of -infinity bounds for a lower bound is extreme -- a more likely value would be zero.
That is,

- individuals who answered NN should be treated as answering Y to a zero-bid value, but NO to the low bid value.

    They should be treated as NY responses -- NO to the lowest bid they were actually presented with, but YES to a bid of 0 (even though they weren't actually asked to respond to a zero bid).
- You might want to use total income as an upper bid value -- *that individuals answering YY would answer NO to a bid value equal to their entire income, but YES to the upper bid value.*

For YN and NY responses provided using the eXplicit TYPE:

When X (explicit) `CHOICE TYPE` is used, you must specify actual values for the upper and lower bounds. However, some respondents may have been YY respondents (or NN), in which case the upper (lower) bound with arbitrary "placeholder values" (say, 100000 or 0) used for the upper and lower bounds.

In this case, when "placeholder" values with no real meaning are sometimes used, you can use `BOUND` to convert these observations to YY (or NN) observations -- essentially, you recode a high (low) placeholder value to infinity (-infinity)

`BOUND` is used to impose these bounds. You can also use the `NEVERYES` option (described) below along with `BOUND` -- this allows you to impose two kinds of bounds (on different classes of respondents).

`NEVERYES` is used to identify non-participants.

There may be respondents who would never say YES, regardless of how low the bid is. In particular, even if the bid was 0, they would not say YES.

These respondents may be different from respondents who say NO to the lowest bid – such respondents may say YES at slightly higher then zero.

Basically, observations identifed as non-participants are treated as NN responses, with a high bid of 0, and a low bid of –infinity.

Using `NEVERYES` with `BOUND`.

`NEVERYES` is designed to be used along with `BOUND`.

First, `BOUND` is used to set the upper and lower bounds (and the type of response). Then `NEVERYES` is used to identify non-particpantes..

For example:

```
BOUND LOW_IND=DOCARE LOW_VAL=0
```

```
        UP_VAR=INCOME ;
    NEVERYES VAR=DONTCARE  ;
```

means:

For respondents who do care, but answered N (or NN), use:
- Lower bound: 0
- Upper bound: lowest bid value,

For respondents who do care, and answered Y (or YY), use:
- Lower bound: highest bid value
- Upper bound: infinity

For respondents who do NOT care (their response is implicitily NN), use:
- Lower bound: -infinity
- Upper bound: 0

Actually, the actual response of non-participants is ignored -- DISCRETE assumes that non-participants must answer N (or NN), and ignores evidence to the contrary!

Alternative -- you can use the same variable in BOUND and in NEVERYES:

```
    BOUND LOW_IND=DONTCARE low_ind_vals=0 LOW_VAL=0
          UP_VAR=INCOME ;
```

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7. Computing Willingness To Pay: with Krinsky Robb and JackKnife Confidence Intervals

DISCRETE can estimate the willingness to pay using a variety of measures.

These include:

Linear	$X\beta$
Integrated expectation, WTP>0	$\int_0^\infty 1 - F(s, X\beta, \sigma) ds$ where F(.) is a CDF depending on the model: PROBIT: $\Phi\left(s - X\beta \big/ \sigma \right)$ LOGIT : $1 \big/ 1 + e^{-(s - X\beta)/\sigma}$ WEIBIT: $1 - \exp\left(- e^{(s - X\beta)/\sigma} \right)$
Integrated expectation, WTP>0 or <0	$\int_0^\infty 1 - F(s, X\beta, \sigma) ds$ - $\int_{-\infty}^0 F(s, X\beta, \sigma) ds$
Truncated TOBIT	$X\beta + \sigma * \phi(X\beta/\sigma) / \Phi(X\beta/\sigma)$ (computed for PROBIT models)
Censored TOBIT	$\Phi(X\beta/\sigma) * X\beta + \sigma\ \phi(X\beta/\sigma)$ (computed for PROBIT models)
Truncated	$X\beta + \rho * \phi(X_1 \beta_1) / \Phi(X_1 \beta_1)$

HECKIT	(computed for 2-stage PROBIT)
Censored HECKIT	$X \beta + \rho * \phi(X_1 \beta_1) / \Phi(X_1 \beta_1)$ (computed for 2-stage PROBIT)
Truncated LOGISTIC	$\sigma * \ln(1 + \exp(X \beta/\sigma))$ (computed for LOGIT)

Where

X: independent variables
β : estimated coefficients
Φ : CDF of standard normal
ϕ : PDF of standard normal
σ: Standard deviation of the normal, or scale parameter of the logistic and "weibull" distribution.
ρ : Correlation between first and second stage error terms

You can also use Krinsky Robb (KR) techniques to compute confidence intervals for the WTP measures. The KR technique works by pulling a new coefficient vector from a multi-variate normal distribution, using the estimated coefficients and the covariance matrix as parameters to the multivariate normal

Or, you can use JackKnife techniques to compute confidence intervals. The JackKnife technique works by creating new datasest by resampling, with replacement, from the observations. For each new dataset, a coefficient vector is estimated.

For further details, see DISCRETE.TXT.

Appendix: Gradient and Hessian of double bounded Logit model

The log-likelihood of double bounded logit model has gradients:

$d \ln(P_y)/d\beta$	$X_y P_y$
$d \ln(P_n)/d\beta$ N	$-X_n(1-P_n)$
$d \ln(P_{yn})/d\beta$	$-X_y P_y(1-P_y) + X_n P_n(1-P_n) \Big/ P_y - P_n$

and a hessian:

$d^2 \ln(P_y)/d\beta^2$	$-X_y\left(X_y P_y(1-P_y)\right)$

$d^2 \ln(P_n)/d\beta^2$	$-X_n\left(X_n P_n (1-P_n)\right)$
$d^2 \ln(P_{yn})/d\beta^2$	$\left(-X_y\left[\left(-d_y P_y\right)+\left(d_y(1-P_y)\right)\right]\right)+\left(-X_n\left[\left(-d_n P_n\right)+\left(d_n(1-P_n)\right)\right]\right) \big/ \left(P_y-P_n\right)$ - $\left[-X_y P_y(1-P_y)+X_n P_n(1-P_n)\right]\left(d_y-d_n\right) \big/ \left(P_y-P_n\right)^2$ Where: $d_y=-X_y\ (1-P_y)\ P_y$ and $d_n=-X_n\ (1-P_n)\ P_n$

The above is somewhat simplified, it abstracts from the vector nature of X (transpose multiplication is required to compute the KxK hessian matrix).

Note that for the single bounded model, just discard the P_{yn} terms.